On the motion of a non-conducting body through a perfectly conducting fluid

By K. STEWARTSON
Department of Mathematics, Durham University, Durham

(Received 24 September 1959)

The motion of bodies in a direction parallel to an applied magnetic field and through a perfectly conducting fluid is considered. It is shown that the perturbation in the state of the fluid cannot remain small except in the particular case when the velocity U of the body is much smaller than that of the Alfvén waves in the fluid. In this case, however, the perturbation is not confined to the neighbourhood of the body, and extends to infinity inside planes which touch the body and are parallel to the undisturbed magnetic field. In addition the body experiences a drag.

1. Introduction

In a previous paper (Stewartson 1956) the motion of a sphere through a conducting fluid in the presence of a strong magnetic field was considered, attention being focused on the ultimate flow pattern assuming that it is steady. It was found that the magnetic field is completely undisturbed but that the velocity field is cylindrical, being independent of the co-ordinate in the direction of the magnetic field. Further, the cylinder C circumscribing the sphere and having its generators parallel to the magnetic field separates two regions with different flow properties: for example, inside C the fluid moves with the sphere as if solid.

The chief assumptions in this theory are that the sphere is a perfect conductor, that the fluid velocity is small compared with the Alfvén velocity and that the magnetic Reynolds number is small. In particular the theory could not apply to a perfectly conducting fluid. The reason for this limitation is that if both the body and the fluid are perfect conductors there is nothing to prevent the build-up of current in the fluid until the magnetic field is seriously perturbed; accordingly, the implicit assumption of small disturbances in the magnetic field is invalidated.

If, however, the *body* is a non-conductor no such difficulty arises for it radiates Alfvén waves into the fluid which serve to control the build-up of current and, in all probability, to keep the disturbances small. It is possible therefore to apply the general method developed earlier to these problems, which are of interest at the present time.

In the present paper we shall make a start on this programme concentrating attention solely on two-dimensional flows in an aligned magnetic field, i.e. the direction of motion of the body relative to the fluid at infinity is parallel to the magnetic field at infinity. First, in § 2 the equations of motion of a perfectly conducting fluid are reduced to a single equation for the stream function ψ

of the motion but which contains two arbitrary functions of ψ. If these functions are constants, ψ satisfies Laplace's equation and a theory of thin aerofoils based on this equation has been developed by Sears & Resler (1959). This theory is discussed in the light of the present work in § 7.

In § 3 the boundary conditions are set out, particular attention being paid to the circumstances in which it is legitimate to require continuity of the tangential component of the magnetic field across the surface of the body. The general unsteady motion of the fluid around a thin aerofoil is analysed in § 4 on the assumption that the disturbances are small. It is shown that if ultimately the flow becomes steady and the perturbations from the undisturbed state are small, the effect of the body on the fluid is non-vanishing at infinity in general. However, on applying the general properties of the ultimate steady flow to determine the flow pattern round a thin aerofoil a contradiction is found because the magnetic field inside the body is seriously affected and with it the velocity field ahead and behind the body. Accordingly, we must conclude that the perturbations if ultimately steady cannot have remained small. There remains, however, one special case, that of a strong magnetic field in which the fluid velocity is in general much smaller than the Alfvén velocity. It turns out that the small perturbation theory can be applied here without contradiction because the fluid velocity and not merely its perturbation component may be assumed small. In § 6 this problem is considered and a full solution obtained for a circular cylinder; other bodies, including thin aerofoils, may be treated if desired. It is concluded that the magnetic field is only slightly disturbed except at two points, but that the magnetic field at infinity is no longer uniform. Further, the fluid inside the planes, which touch the cylinder and are parallel to the undisturbed magnetic field, moves with the cylinder as if solid; the fluid outside them is at rest. It is noted that many of the features of the motions described earlier (Stewartson 1956) are repeated here: the differences may be ascribed to the finite conductivity of the fluid in the earlier problem.

2. Equations of motion

The equations governing the motion of an incompressible fluid of density ρ, permeability unity, electrical conductivity σ and kinematic viscosity ν are

$$\operatorname{div}\mathbf{q} = 0, \tag{2.1}$$

$$\frac{\partial \mathbf{q}}{\partial t} - \mathbf{q}\wedge\operatorname{curl}\mathbf{q} = -\operatorname{grad}\left(\frac{p}{\rho}+\tfrac{1}{2}\mathbf{q}^2\right)+\frac{1}{4\pi\rho}(\operatorname{curl}\mathbf{H}\wedge\mathbf{H})+\nu\nabla^2\mathbf{q}, \tag{2.2}$$

$$\frac{\partial \mathbf{H}}{\partial t} - \operatorname{curl}(\mathbf{q}\wedge\mathbf{H}) = \frac{1}{4\pi\sigma}\nabla^2\mathbf{H}, \tag{2.3}$$

where $\mathbf{q}$ is the velocity of the fluid, $\mathbf{H}$ the magnetic field and p the pressure. The units are Gaussian. In these equations it is assumed that the effects of the flow of charged particles and of the temporal variation of the displacement vector on the current are negligible. Once the initial uniform magnetic field has been set up, these assumptions can be justified in the types of flow considered here.

Consider the particular case of the steady two-dimensional flow of a perfectly conducting inviscid fluid. If, in addition, the velocity of the fluid and the magnetic field are parallel at infinity, equations (2.1)–(2.3) may be formally simplified. For, using an orthogonal Cartesian triad *Oxyz* as frame of reference and taking $\mathbf{q}$, $\mathbf{H}$ to be parallel to the plane $z = 0$, we may write, from (2.1),

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}, \tag{2.4}$$

where $\mathbf{q} = (u, v, 0)$ and ψ is a stream function. Then from (2.3), since $\sigma = \infty$,

$$\mathbf{q} \wedge \mathbf{H} = A\mathbf{k},$$

where $\mathbf{k}$ is a unit vector parallel to the z-axis and A is a constant scalar. Further, since $\mathbf{q}$ is parallel to $\mathbf{H}$ at infinity, $A = 0$, whence

$$\mathbf{H} = \alpha \mathbf{q}. \tag{2.5}$$

Here α is a scalar function of position, which may be shown to be a function of ψ only by the use of the equation $\operatorname{div} \mathbf{H} = 0$. On substituting into (2.2) and setting $\nu = 0$, we obtain

$$-\mathbf{q} \wedge \operatorname{curl} \mathbf{q} = -\operatorname{grad}\left\{\frac{p}{\rho} + \tfrac{1}{2}\mathbf{q}^2\right\} + \frac{\alpha(\psi)}{4\pi\rho}\{\alpha(\psi) \operatorname{curl} \mathbf{q} - \alpha'(\psi)\, \mathbf{q} \wedge \operatorname{grad} \psi\} \wedge \mathbf{q},$$

i.e.

$$\left\{\left(1 - \frac{\alpha^2}{4\pi\rho}\right)\nabla^2\psi - \frac{\alpha\alpha'}{4\pi\rho}\mathbf{q}^2\right\}\operatorname{grad}\psi = \operatorname{grad}\left(\frac{p}{\rho} + \tfrac{1}{2}\mathbf{q}^2\right),$$

so that

$$\left(1 - \frac{\alpha^2}{4\pi\rho}\right)\nabla^2\psi = \frac{\alpha\alpha'}{4\pi\rho}(\operatorname{grad}\psi)^2 + \beta(\psi), \tag{2.6}$$

where $\beta(\psi)$, like $\alpha(\psi)$, is an arbitrary function of ψ which must be determined before (2.6) can be solved for ψ.

Suppose now that at infinity upstream the velocity is uniform with components $(U, 0, 0)$ and that the magnetic field is uniform with components $(H_\infty, 0, 0)$. Then

$$\alpha(\psi) = H_\infty/U, \quad \beta(\psi) = 0, \tag{2.7}$$

since $\psi \to Uy$ as $x \to \infty$, and hence

$$\nabla^2\psi = 0 \tag{2.8}$$

on all streamlines starting from an infinite distance upstream. Then if we can be sure that no streamline in the flow is closed and that none starts and ends at an infinite distance downstream, we may conclude that the motion of the fluid is exactly the same as when the magnetic field is absent and that the magnetic lines of force coincide with the streamlines. In addition as may be seen from the next section, the boundary conditions on a fixed solid body, consisting of non-magnetic insulating material, are all satisfied because the magnetic field is tangential to it on its surface.

However, we have no guarantee that in any steady motion which can be set up from rest $\alpha(\psi)$ must be constant, nor can we be sure that every streamline in the flow field starts from an infinite distance upstream. If the fluid is set into motion with constant velocity $(U, 0, 0)$ at infinity and moves past a fixed obstacle, the velocity field at $t = 0+$ will certainly be irrotational. If in addition a uniform magnetic field $(H_\infty, 0, 0)$ has been applied, at infinity the magnetic field will still be uniform at $t = 0+$. Hence, $\text{curl}(\mathbf{q} \wedge \mathbf{H}) \neq 0$ at $t = 0+$ so that $\mathbf{H}$ must begin to change. Part of this change is accounted for by the convection of the magnetic field by the fluid and part by the Alfvén waves set up to ensure that the new magnetic field satisfies the boundary conditions at the body. These waves start at the body and move relative to the fluid with a velocity

$$\frac{|\mathbf{H}|}{(4\pi\rho)^{\frac{1}{2}}}. \tag{2.9}$$

Accordingly downstream of the body these waves can be expected to reach infinity where, unless their strength has declined to infinitesimal proportions, they will modify the flow and disturb the values of α, β assumed in (2.7). It is conceivable, for example, that they could cause a reversed flow behind the body. A similar argument holds upstream of the body except that the absolute velocity of the Alfvén waves is reduced, since they are being propagated against the direction of motion of the fluid. If, however,

$$U < \frac{H_\infty}{(4\pi\rho)^{\frac{1}{2}}}, \tag{2.10}$$

we can expect them to reach infinity upstream and again they may distort the values of α, β leading to a rotational motion of the fluid everywhere. On the other hand, if

$$U > \frac{H_\infty}{(4\pi\rho)^{\frac{1}{2}}}, \tag{2.11}$$

Alfvén waves can only penetrate to infinity upstream by a more indirect process, since in the region where the flow is almost undisturbed their relative velocity is too small to permit any penetration in the upstream direction. Hence at any stage the velocity field ahead of the furthest point hitherto reached by the Alfvén wave must have been seriously weakened by the presence of the body before the waves can penetrate further. This requirement seems to be too stringent and it appears reasonable to conjecture that, on the streamlines which come from infinity upstream, (2.8) holds with (2.11). These streamlines need not cover the whole region outside the body, however, and the possibility of closed streamlines near the forward stagnation point and of reversed flow behind the body cannot be excluded.

In view of the doubts which have been expressed about the validity of using (2.8) to describe the whole field of flow, it is worth discussing in a particular case the way in which the motion develops as t increases and the ultimate motion which is produced as $t \to \infty$. Before doing this, however, we shall discuss the appropriate boundary conditions which are not all immediately obvious.

3. Boundary conditions

We consider the problem of a fixed body in a perfectly conducting inviscid fluid which at time $t = 0$ is given a velocity U at infinity in the direction of x increasing. At the same or earlier time a magnetic field of magnitude H_∞ is imposed in the same direction. The following boundary conditions can be written down at once. First, at $t = 0+$ the velocity field is the irrotational field which would be appropriate to this problem if there were no magnetic field. The magnetic field is still uniform and of magnitude H_∞ at $t = 0+$. Secondly, at an infinite distance from the body

$$\mathbf{q} \to (U, 0, 0), \quad \mathbf{H} \to (H_\infty, 0, 0) \tag{3.1}$$

at any *finite* time t, since all disturbances start from the body. This does *not* imply that (3.1) holds at an infinite time.

In order to ascertain the correct boundary conditions at the body some care is needed. We shall suppose that the body is composed of a non-magnetic, non-conducting material so that it is not a source of magnetic field and in it $\sigma = 0$. Two conditions at the surface of the body, namely, that the normal components of the magnetic field and of the velocity must be continuous and zero, respectively, are clear either from the equation of continuity or on physical grounds. If the normal component of the magnetic field vanishes on the body then there is no restriction on the tangential components of the magnetic field, the reason being that surface currents cannot be dispersed. In general, however, we cannot assume that the normal component of the magnetic field is zero at the body and we shall obtain the conditions on the tangential components when it does not vanish.

If the boundary of a perfectly conducting fluid is free it has already been shown (Stewartson 1957) that the tangential component of the magnetic field must be continuous. An initial forced discontinuity is immediately dispersed, travelling into the fluid as an Alfvén wave. This result was proved by considering an inviscid fluid of large but finite conductivity σ and proceeding to the limit $\sigma \to \infty$. There was no need to assume that the fluid had a small coefficient of kinematic viscosity ν and to allow $\nu \to 0$ independently because viscous effects in this problem are of the second order. The reason is that the adjustment of the velocity field to preserve a continuous tangential component of stress at the boundary is vanishingly small as $\nu \to 0$. At a solid boundary, however, viscous effects may matter because the appropriate condition is that $\mathbf{q} = 0$ at the boundary. Hence as $\nu \to 0$ a boundary layer in $\mathbf{q}$ must develop which may have serious effects on the magnetic field. The mutual effects of the velocity and magnetic fields can be determined from the following problem of steady motion.

A fluid of conductivity σ and kinematic viscosity ν is in contact with a fixed solid insulator along the plane $x = 0$. In the solid, which occupies the region $x < 0$, the magnetic field is $(H_0, H_-, 0)$ and in the fluid the magnetic field is $(H_0, h, 0)$ and the velocity field is $(0, v, 0)$. In the fluid at large distances from the solid $v \to U$, $h \to H_+$. The motion is one-dimensional so that v, h are functions of x only; all other quantities being constant. The equations governing the motion of the fluid, (2.1)–(2.3), then reduce to

$$-H_0 \frac{\partial v}{\partial x} = \frac{1}{4\pi\sigma}\frac{\partial^2 h}{\partial x^2}, \quad -\frac{H_0}{4\pi\rho}\frac{\partial h}{\partial x} = \nu \frac{\partial^2 v}{\partial x^2}. \tag{3.2}$$

The appropriate boundary conditions are that

$$\left.\begin{aligned} v \to U, \quad h \to H_+ \quad &\text{as} \quad x \to \infty +, \\ v = 0, \quad h = H_- \quad &\text{as} \quad x \to 0, \end{aligned}\right\} \tag{3.3}$$

since for a fluid of *finite* conductivity all components of the magnetic field are continuous at its boundary. The solution of (3.2) subject to (3.3) is

$$\left. \begin{aligned} h &= H_+ + (H_- - H_+)\exp\left\{-H_0 x\left(\frac{\sigma}{\rho\nu}\right)^{\frac{1}{2}}\right\}, \\ v &= \frac{H_+ - H_-}{4\pi(\rho\nu\sigma)^{\frac{1}{2}}}\left[1 - \exp\left\{-H_0 x\left(\frac{\sigma}{\rho\nu}\right)^{\frac{1}{2}}\right\}\right]. \end{aligned} \right\} \tag{3.4}$$

Hence

$$U = \frac{H_+ - H_-}{4\pi(\rho\nu\sigma)^{\frac{1}{2}}} \tag{3.5}$$

and is not arbitrary. Now if we let $\nu \to 0$, $\sigma \to \infty$ to obtain a perfectly conducting inviscid fluid it follows that

$$h = H_-, \quad v = 0 \quad \text{if} \quad x < 0;$$

$$h = H_+, \quad v = (H_+ - H_-)/4\pi(\rho\nu\sigma)^{\frac{1}{2}} \quad \text{if} \quad x > 0,$$

a magnetohydrodynamic boundary layer of a particularly simple kind being formed at $x = 0$. Thus in this specific problem it is not necessary that the tangential component of $\mathbf{q}$ should be zero at the boundary or that the tangential components of $\mathbf{H}$ must be continuous in the limit $\nu \to 0$, $\sigma \to \infty$. We must, however, have

$$[\mathbf{H}]_s = [\mathbf{q}]_s 4\pi(\rho\nu\sigma)^{\frac{1}{2}}, \tag{3.6}$$

where $[\;]_s$ denotes 'the leap in the tangential component of'. The value of $\nu\sigma$ for a perfectly conducting inviscid fluid is indeterminate, but for a real fluid which is highly conducting and almost inviscid it can be worked out from the known physical properties. Actually in fluids of importance in terrestrial and interplanetary experiments $\nu\sigma$ is invariably very small and so in this paper we shall replace (3.6) by

$$[\mathbf{H}]_s = 0, \tag{3.7}$$

but there is no special difficulty about using (3.6).

Although (3.6) has been deduced for one specific and simple case the generalization to a curved wall and a general magnetohydrodynamic field is immediate. For, when σ is large and ν small the changes implied in (3.4) occur so rapidly in the x-direction that other changes due to wall curvature, etc., are negligible. The relevant equations governing the magnetohydrodynamic boundary layer at the wall are therefore identical with (3.2) and we conclude that (3.6) is a general condition on $\mathbf{q}$, $\mathbf{H}$ which must be satisfied at the boundary of a solid insulator with a perfectly conducting inviscid fluid.

If initially the motion is such that (3.6) is *not* satisfied then the manner in which the appropriate discontinuities are dispersed may easily be investigated and follows the same lines as when the boundary of the fluid is free. It will not be given here; it is merely pointed out that just as in the case of a free boundary the dispersion is immediate. The interested reader can readily work out the details for himself using the arguments of Stewartson (1957) and the corresponding equations allowing for viscosity. These equations have been discussed by Ludford (1959) in a related problem of magnetohydrodynamics.

4. Unsteady perturbations in a perfectly conducting fluid

In this section we consider the perturbation induced by a thin body placed along the x-axis, assuming that at time $t < 0$ the magnetic field is $(H_\infty, 0, 0)$, and that the fluid is set in motion at $t = 0$ with a velocity $(U, 0, 0)$ at infinity. Since the body is thin we can write

$$\mathbf{q} = (U+u, v, 0), \quad \mathbf{H} = (H_\infty + h_x,\ h_y,\ 0), \tag{4.1}$$

where u, v, h_x, h_y are all small. Neglecting squares and products of these small quantities and setting $\nu = 0$, $\sigma = \infty$, equations (2.1)–(2.3) reduce to

$$\frac{\partial h_x}{\partial x} + \frac{\partial h_y}{\partial y} = 0, \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{4.2}$$

$$\left.\begin{aligned} &\frac{\partial h_x}{\partial t} + U\frac{\partial h_x}{\partial x} = H_\infty \frac{\partial u}{\partial x}, && \frac{\partial h_y}{\partial t} + U\frac{\partial h_y}{\partial x} = H_\infty \frac{\partial v}{\partial x}, \\ &\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} = \frac{\partial P}{\partial x} + \frac{H_\infty}{4\pi\rho}\frac{\partial h_x}{\partial x}, && \frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x} = \frac{\partial P}{\partial y} + \frac{H_\infty}{4\pi\rho}\frac{\partial h_y}{\partial x}, \end{aligned}\right\} \tag{4.3}$$

where
$$P = -\frac{p}{\rho} - \frac{H_\infty^2}{4\pi\rho} + \text{const.} \tag{4.4}$$

These equations may be divided into two parts. First, there is a part which is directly due to P and which may easily be seen to be irrotational. Denoting its contributions to $\mathbf{q}$, $\mathbf{H}$ by a superscript (1) and writing

$$h_x^{(1)} = H_\infty \frac{\partial^2 \phi}{\partial x^2}, \quad h_y^{(1)} = H_\infty \frac{\partial^2 \phi}{\partial x\, \partial y}, \quad \text{where} \quad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, \tag{4.5}$$

it follows that
$$u^{(1)} = \frac{\partial^2 \phi}{\partial t\, \partial x} + U\frac{\partial^2 \phi}{\partial x^2}, \quad v^{(1)} = \frac{\partial^2 \phi}{\partial t\, \partial y} + U\frac{\partial^2 \phi}{\partial x\, \partial y}, \tag{4.6}$$

$$P = \frac{\partial^2 \phi}{\partial t^2} + 2U\frac{\partial^2 \phi}{\partial t\, \partial x} + \left(U^2 - \frac{H_\infty^2}{4\pi\rho}\right)\frac{\partial^2 \phi}{\partial x^2}. \tag{4.7}$$

The second part is explicitly independent of P and is wave-like. Denoting its contributions to $\mathbf{q}$, $\mathbf{H}$ by a superscript (2)

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2 u^{(2)} = \frac{H_\infty^2}{4\pi\rho}\frac{\partial^2 u^{(2)}}{\partial x^2}, \quad \frac{H_\infty}{4\pi\rho}\frac{\partial h_x^{(2)}}{\partial x} = \frac{\partial u^{(2)}}{\partial t} + U\frac{\partial u^{(2)}}{\partial x} \tag{4.8}$$

with corresponding equations for $v^{(2)}$, $h_y^{(2)}$, the two pairs of functions being connected by (4.2). These solutions correspond to the Alfvén waves mentioned in § 2 and we see that in this small disturbance theory the irrotational and wave-like motions are additive.

The solution of (4.8) for $u^{(2)}$ is

$$\frac{u^{(2)}}{U} = f\left(t - \frac{mx}{(1+m)U}, y\right) + g\left(t + \frac{mx}{(1-m)U}, y\right), \tag{4.9}$$

where f, g are arbitrary functions and

$$H_\infty m = U(4\pi\rho)^{\frac{1}{2}},$$

m being the ratio of the velocity of the undisturbed fluid to the velocity of the Alfvén waves. All disturbances to the motion originate at the body, none come from infinity, and hence if $m < 1$

$$u^{(2)} = Uf\left\{t - \frac{mx}{(1+m)\,U}, y\right\}, \quad h_x^{(2)} = -mH_\infty f \quad \text{when} \quad x > 0, \tag{4.10}$$

$$u^{(2)} = Ug\left\{t + \frac{mx}{(1-m)\,U}, y\right\}, \quad h_x^{(2)} = mH_\infty g \quad \text{when} \quad x < 0, \tag{4.11}$$

while if $m > 1$

$$u^{(2)} = U(f+g), \quad h_x^{(2)} = mH_\infty(g-f) \quad \text{when} \quad x > 0, \tag{4.12}$$

$$u^{(2)} = h_x^{(2)} = 0 \quad \text{when} \quad x < 0, \tag{4.13}$$

from which $v^{(2)}$, $h_y^{(2)}$ may be written down.

In particular as $t \to \infty$ these functions become independent of t if, as we shall assume, the motion is ultimately steady. Hence writing

$$f(\infty, y) = f(y), \quad g(\infty, y) = g(y), \tag{4.14}$$

we have if $m < 1$

$$\left.\begin{aligned} u^{(2)} &= Uf(y), \quad h_x^{(2)} = -mH_\infty f(y), \quad v^{(2)} = h_y^{(2)} = 0 \quad (x > 0),\\ u^{(2)} &= Ug(y), \quad h_x^{(2)} = mH_\infty g(y), \quad v^{(2)} = h_y^{(2)} = 0 \quad (x < 0),\end{aligned}\right\} \tag{4.15}$$

while if $m > 1$

$$\left.\begin{aligned} &u^{(2)} = U\{f(y)+g(y)\}, \quad h_x^{(2)} = mH_\infty\{g(y)-f(y)\}, \quad v^{(2)} = h_y^{(2)} = 0, \quad \text{for} \quad x > 0,\\ &u^{(2)} = v^{(2)} = h_x^{(2)} = h_y^{(2)} = 0 \quad \text{for} \quad x < 0.\end{aligned}\right\} \tag{4.16}$$

Hence, if the disturbances are always small, it is given, if the ultimate motion is steady, by the sum of (4.5), (4.6) and (4.15) or (4.16), viz.

$$\left.\begin{aligned} u &= U\left(\frac{\partial^2\phi}{\partial x^2} + f(y)\right), \quad v = U\frac{\partial^2\phi}{\partial x\,\partial y},\\ h_x &= H_\infty\left(\frac{\partial^2\phi}{\partial x^2} - mf(y)\right), \quad h_y = H_\infty\frac{\partial^2\phi}{\partial x\,\partial y} \quad \text{for} \quad x > 0,\end{aligned}\right\} \tag{4.17}$$

$$\left.\begin{aligned} u &= U\left(\frac{\partial^2\phi}{\partial x^2} + g(y)\right), \quad v = U\frac{\partial^2\phi}{\partial x\,\partial y},\\ h_x &= H_\infty\left(\frac{\partial^2\phi}{\partial x^2} + mg(y)\right), \quad h_y = H_\infty\frac{\partial^2\phi}{\partial x\,\partial y} \quad \text{for} \quad x < 0,\end{aligned}\right\} \tag{4.18}$$

if $m < 1$; and

$$\left.\begin{aligned} u &= U\left(\frac{\partial^2\phi}{\partial x^2} + f(y) + g(y)\right), \quad v = U\frac{\partial^2\phi}{\partial x\,\partial y},\\ h_x &= H_\infty\left\{\frac{\partial^2\phi}{\partial x^2} + mg(y) - mf(y)\right\}, \quad h_y = H_\infty\frac{\partial^2\phi}{\partial x\,\partial y} \quad \text{for} \quad x > 0,\end{aligned}\right\} \tag{4.19}$$

$$u = U\frac{\partial^2\phi}{\partial x^2}, \quad v = U\frac{\partial^2\phi}{\partial x\,\partial y}, \quad h_x = H_\infty\frac{\partial^2\phi}{\partial x^2}, \quad h_y = H_\infty\frac{\partial^2\phi}{\partial x\,\partial y} \quad \text{for} \quad x < 0, \tag{4.20}$$

if $m > 1$, where ϕ is harmonic and f, g are arbitrary functions of y. Inside the body the magnetic field is harmonic since it is non-magnetic and an insulator. Hence we may write

$$h_x = H_\infty \frac{\partial^2 \psi}{\partial x^2}, \quad h_y = H_\infty \frac{\partial^2 \psi}{\partial y \partial x}, \tag{4.21}$$

where ψ is harmonic and analytic inside the body.

5. Thin aerofoils in steady motion

Let us now apply the theory of the previous section to the flow engendered by a slender convex aerofoil, symmetrically disposed with respect to the x-axis and whose surface is given by

$$|y| = \epsilon S(x) \quad (-b \leqslant x \leqslant a), \tag{5.1}$$

where ϵ is small and S is a function of x of order one, with a bounded derivative and vanishing at $x = -b, a$. Then the boundary conditions to be satisfied imply that on the aerofoil

$$v = \epsilon U S'(x) \operatorname{sgn} y, \tag{5.2}$$

whence, from (4.17)–(4.20),

$$\frac{\partial \phi}{\partial x} = \frac{\epsilon}{2\pi} \int_{-b}^{a} S'(x_1) \log\{(x-x_1)^2 + y^2\}\, dx_1 \tag{5.3}$$

since ϕ is harmonic, where a prime denotes differentiation. Further, in order to ensure that $\mathbf{q}$, $\mathbf{H}$ are continuous in the fluid the plane $x = 0$ must be chosen to pass through the aerofoil at its maximum cross-section, and then

$$f(y) = g(y) = 0$$

if $|y| > \epsilon S(0)$.

Since $Uh_y = H_\infty v$ outside the aerofoil

$$h_y = \epsilon H_\infty S'(x) \operatorname{sgn} y \tag{5.4}$$

on the aerofoil; hence to a first approximation

$$h_y = \frac{H_\infty y S'(x)}{S(x)} \tag{5.5}$$

inside the aerofoil. Using the equation of continuity and (5.5) it follows that

$$h_x = H_\infty \log S(x) + \text{const.} \tag{5.6}$$

inside the aerofoil to a first approximation in ϵ. The additive constant is apparently indeterminate in slender wing theory. Thus although h_y is small in the aerofoil h_x is of the same order as H_∞ so that the field inside is seriously perturbed.*

Since

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{\epsilon}{2\pi} \int_{-b}^{a} \frac{S'(x_1)\, dx_1}{x - x_1} \tag{5.7}$$

* In fact it has a logarithmic singularity at $S(x) = 0$ but it must be remembered that singularities at isolated points are often removable in aerofoil theory: if this were the only objection to the present argument it would be felt that the theory were substantially valid.

on the aerofoil, it follows that

$$mg(y) = -\frac{\epsilon}{2\pi}\int_{-b}^{a}\frac{S'(x_1)\,dx_1}{x-x_1}+\log S(x)+\text{const.}\quad(-b < x < 0), \tag{5.8}$$

where $y = \epsilon S(x)$, with a similar formula for $f(y)$. Thus if $m < 1$ the magnetic field and the velocity field are seriously distorted inside the planes $y = \pm\epsilon S(0)$, and since this region is the most vital part of the fluid the initial assumption that the disturbances remain small must be regarded as incorrect. If $m > 1$ the situation is even more serious for (5.6) contradicts (4.20) and so no solution is possible at all. A study of the initial motion of the fluid past the aerofoil from a state of rest shows that the motion in which the disturbances are small in fact breaks down straight away. The reason is that ahead of the body, according to small disturbance theory, no Alfvén waves can occur. Hence, on that part of the boundary for which $x < 0$ the continuity of h_x, h_y implies that the magnetic field inside the aerofoil is given by the analytic continuation of ϕ. But at $t = 0+$, ϕ is due to a source distribution in the aerofoil along the x-axis and hence so is the magnetic field. This is a contradiction, since the aerofoil is non-magnetic. It should be noted that the difficulties in the theory when $m > 1$ are due entirely to the linearization of the equations. When the full equations are used, Alfvén waves have no difficulty propagating into the fluid from the aerofoil. If the more general boundary condition (3.6) is used instead of (3.7) the theory also breaks down. The reason is that any restriction on the tangential component of $\mathbf{q}$ implies that the value of v at the boundary is not small in comparison with U.

The main conclusion to be drawn from the work of this section may be stated in the following way. Let a thin non-conducting aerofoil be set in motion, with uniform velocity $U \neq 0$ in a direction parallel to its length through a perfectly conducting fluid; then the perturbation in the state of the fluid caused by the motion of the aerofoil cannot ultimately become steady if it remains small.

There is one exception, however. If $m \ll 1$ it is possible to obtain a consistent linearized theory of the motion because the fluid velocity and not merely its perturbation component may be assumed small. The ultimate motion in this case is discussed in the next section.

6. The slow motion of a body in a perfectly conducting fluid

The equations which govern the state of the fluid in this limiting problem are the same as (4.2)–(4.4), provided we neglect the terms in them containing U as a multiplicative factor. It is assumed until proved otherwise that the disturbances remain small; it is worth noting, however, that it is no longer necessary to restrict the aerofoil to be thin. If the theory is valid at all it is valid for all bodies. The general solution of the equations is now

(*a*) For $x > 0$,

$$\left.\begin{aligned} h_x &= H_\infty\frac{\partial^2\phi}{\partial x^2} - U(4\pi\rho)^{\frac{1}{2}} f\left(t-\frac{x}{V}, y\right),\\ u &= \frac{\partial^2\phi}{\partial t\,\partial x} + Uf\left(t-\frac{x}{V}, y\right). \end{aligned}\right\} \tag{6.1}$$

(*b*) For $x < 0$,

$$\left.\begin{aligned} h_x &= H_\infty \frac{\partial^2 \phi}{\partial x^2} + U(4\pi\rho)^{\frac{1}{2}} g\left(t + \frac{x}{V}, y\right), \\ u &= \frac{\partial^2 \phi}{\partial t \partial x} + U g\left(t + \frac{x}{V}, y\right), \end{aligned}\right\} \tag{6.2}$$

where ϕ is harmonic and of the same order as U, while

$$V = \frac{H_\infty}{(4\pi\rho)^{\frac{1}{2}}}. \tag{6.3}$$

From (6.1) and (6.2), v and h_y may be written down. In particular, if the motion of the fluid is ultimately steady we have, using (4.14),

$$u = Uf(y), \quad v = 0, \quad h_x = H_\infty \frac{\partial^2 \phi}{\partial x^2} - U(4\pi\rho)^{\frac{1}{2}} f(y), \quad h_y = H_\infty \frac{\partial^2 \phi}{\partial x \partial y} \tag{6.4}$$

if $x > 0$; and

$$u = Ug(y), \quad v = 0, \quad h_x = H_\infty \frac{\partial^2 \phi}{\partial x^2} + U(4\pi\rho)^{\frac{1}{2}} g(y), \quad h_y = H_\infty \frac{\partial^2 \phi}{\partial x \partial y} \tag{6.5}$$

if $x < 0$. Further
$$P = \text{const.} - \frac{H_\infty}{4\pi\rho}\left(h_x - H_\infty \frac{\partial^2 \phi}{\partial x^2}\right). \tag{6.6}$$

Suppose now that the body is convex and denote its boundary by S. Then the plane $x = 0$ must contain its maximum cross-section, and if further this cross-section lies between $y = \pm c$ it follows that

$$\left.\begin{aligned} g(y) = f(y) = 1 \quad \text{for} \quad |y| < c, \\ g(y) = f(y) = 0 \quad \text{for} \quad |y| > c, \end{aligned}\right\} \tag{6.7}$$

to satisfy the condition on the velocity at S. According to (6.7) the fluid contained within $|y| = c$ is completely at rest, while outside $|y| = c$ on the other hand it moves with uniform velocity U completely undisturbed by the presence of the body. In order to find the magnetic field, let it be given by

$$h_x = H_\infty \frac{\partial^2 \psi}{\partial x^2}, \quad h_y = H_\infty \frac{\partial^2 \psi}{\partial y^2} \tag{6.8}$$

inside S where ψ is harmonic and regular there. Since all components of $\mathbf{H}$ must be continuous at S, it follows that

$$\frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial x \partial y} \quad \text{at } S, \tag{6.9}$$

$$\left.\begin{aligned} \frac{\partial^2 \psi}{\partial x^2} = \frac{\partial^2 \phi}{\partial x^2} - \frac{U}{V} \quad \text{at } S, \quad \text{for} \quad x > 0, \\ \frac{\partial^2 \psi}{\partial x^2} = \frac{\partial^2 \phi}{\partial x^2} + \frac{U}{V} \quad \text{at } S, \quad \text{for} \quad x < 0. \end{aligned}\right\} \tag{6.10}$$

The detailed properties of ψ, ϕ depend on the explicit form taken by S; as an example we shall now consider the case of a circular cylinder whose boundary is given by

$$r = c, \tag{6.11}$$

where $r\cos\theta = x$, $r\sin\theta = y$.

Let
$$\frac{\partial \phi}{\partial x} = \sum_{n=1}^{\infty} A_n \left(\frac{c}{r}\right)^n \frac{\cos n\theta}{n} - A_0 \log r, \tag{6.12}$$
$$\frac{\partial \psi}{\partial x} = \sum_{n=1}^{\infty} B_n \left(\frac{r}{c}\right)^n \frac{\cos n\theta}{n} + B_0, \tag{6.13}$$
where A_n, B_n are all constants to be found. From (6.9)
$$-\sum_{0}^{\infty} A_n \sin(n+1)\theta = -\sum_{2}^{\infty} B_n \sin(n-1)\theta,$$
i.e.
$$B_n = A_{n-2} \quad (n \geqslant 2), \tag{6.14}$$
whence from (6.10)
$$-2\sum_{n=0}^{\infty} A_n \cos(n+1)\theta - B_1 = \frac{U}{V}\operatorname{sgn}(\cos\theta) = \frac{4U}{\pi V}\sum_{s=0}^{\infty}\frac{(-)^s}{2s+1}\cos(2s+1)\theta,$$
i.e.
$$B_1 = 0, \quad A_{2s+1} = 0, \quad A_{2s} = \frac{2U(-)^{s+1}}{V\pi(2s+1)} \quad (s = 0, 1, 2, \ldots). \tag{6.15}$$
Using (6.15) in conjunction with (6.12), it is a straightforward matter to show that
$$\frac{\partial \phi}{\partial x} = \frac{U}{2\pi V}\left[4c - 2x\tan^{-1}\frac{2cx}{r^2-c^2} + (y-c)\log\{x^2+(y-c)^2\} - (y+c)\log\{x^2+(y+c)^2\}\right], \tag{6.16}$$
while $\partial\psi/\partial x$ differs from $\partial\phi/\partial x$ by a constant only. According to (6.16) the magnetic field has a logarithmic singularity at $x = 0$, $y = \pm c$ and so the assumption of small disturbances breaks down in the neighbourhood of these points. Since U is small the region in which the basic assumption of small disturbances is invalid is exponentially small, and in consequence it is unlikely that the correction when the full equations are used will be significant. In fact the breakdown of the small-disturbance theory in this problem would seem to be of negligible importance when compared with the breakdown in conventional thin aerofoil theory. It should be remembered, however, that that theory, unlike ours, has a solid corpus of exact solutions to support it.

At large distances from the circular cylinder the total magnetic field consists of two parts. First, there is a component in the x-direction equal to
$$H_\infty - U(4\pi\rho)^{\frac{1}{2}}\operatorname{sgn} x, \tag{6.17}$$
if $|y| < c$, and to H_∞ if $|y| > c$. This field does not of course tend to zero as $|x| \to \infty$. Secondly, there is a contribution due to an apparent magnetic pole at the origin of strength
$$\frac{2U}{\pi}(4\pi\rho)^{\frac{1}{2}}. \tag{6.18}$$

The pressure may be calculated from (6.6), and as $t \to \infty$ we find that
$$p \to p_0 - \left(\frac{\rho}{4\pi}\right)^{\frac{1}{2}} UH_\infty \operatorname{sgn} x \quad \text{for} \quad |y| < c,$$
$$p \to p_0 \quad \text{for} \quad |y| > c,$$

where p_0 is a constant. Hence the force on the circular cylinder tends to

$$2c\left(\frac{\rho}{\pi}\right)^{\frac{1}{2}} UH_{\infty} \text{ per unit thickness,} \tag{6.19}$$

in the direction of x increasing. This result is also true for all convex bodies, whatever their shape, provided only that their maximum thickness is $2c$.

A previous paper by the author (1956) was concerned with the motion of a perfectly conducting sphere through an imperfectly conducting fluid with conductivity σ. Although the conclusions as to the ultimate motion of the fluid were the same as those obtained here there are several important points of divergence. First, an important condition of the previous theory is that $\sigma U \ll 1$ so that it is not strictly applicable to a perfectly conducting fluid. Secondly, the condition on the magnetic field at the surface of the sphere is simply that the normal component of the perturbed field should vanish while the tangential component may be discontinuous. This condition can be satisfied without the necessity for any Alfvén waves and in fact as $\sigma \to \infty$ it was shown that the perturbed field increased indefinitely without changing the velocity field. Thus if the body is a perfect conductor, no matter how slowly it moves through a perfectly conducting fluid we can expect the magnetic field to be seriously perturbed ultimately. This is because there is nothing to prevent the build-up of current when U is small. In the present problem the build-up is controlled by the Alfvén waves radiated from the body. Finally, it should be noted that in the earlier problem the magnetic field was ultimately uniform and there was no force on the body.

7. Discussion

The steady motion of a non-conducting body through a perfectly conducting inviscid fluid in the presence of an aligned magnetic field has been shown to be a formidable problem. If one considers only the steady state it is found that the motion of the fluid is indeterminate depending on two arbitrary functions of the stream function ψ. If the conditions a long way upstream and downstream are uniform, then both of these functions can be found, but it must be expected that such a simple state of affairs will not exist in general.

These unknown functions can be found in any particular case by examining how the steady state is set up. For example, this may be done by setting the body in motion and tracing the behaviour of the fluid as $t \to \infty$. In such a method the fluid velocity and the magnetic field are uniquely determinate at all values of t including $t = \infty$, and so we can expect to obtain some information about the unknown functions of ψ by considering the unsteady equations of motion. This can be done fairly easily if the perturbations are small, and we have been able to find the properties of the ultimate flow assuming that the perturbations are small. On applying these properties to the motion of thin aerofoils, however, it is found that the ultimate motion is either not steady or not small, because otherwise large disturbances occur in the magnetic field in the body. One exceptional case, when the applied magnetic field is strong, can be successfully carried through and consistent results are found. Of these the most important is that the fluid inside planes touching the body and parallel to the imposed magnetic field moves with the body as if solid. This result is true for all bodies and not only for thin aerofoils.

These conclusions may be compared with a theory of thin aerofoils in a perfectly conducting fluid recently developed by Sears & Resler (1959). In their paper two cases are considered according as the applied magnetic field is parallel or perpendicular to the direction of motion of the aerofoil, but we are concerned only with the first of these here. In the theory it is assumed that the conditions at infinity both upstream and downstream of the body are undisturbed, in which case the stream function is harmonic exactly as if there were no magnetic field, and the magnetic field is zero inside the body. All the requirements which can be derived from steady state equations and boundary conditions are satisfied. Attractive as this theory is, it is only one of an infinite number of equally self-consistent theories: further the assumption that conditions at infinity are undisturbed, while in accord with the corresponding theory in the absence of a magnetic field, is not in accord with the corresponding theory for a transverse magnetic field in which the effect of the aerofoil extends indefinitely in certain directions: again as $m \to 0$ and the speed of the aerofoil is reduced to zero, the magnetic field is not uniform everywhere since it must be zero inside the body.

None of these features implies that the theory is invalid but they do make it questionable whether it is realistic, i.e. can be produced in an experiment. Certainly if by some means the motion envisaged could be set up then there is absolutely no reason why it should not continue. On the other hand the same is true of an infinity of other solutions. The natural way of setting up a steady motion is to set up the field first and then start the relative motion of body and fluid. Accordingly the steady motion required is preceded by an unsteady motion in which disturbances from the uniform state are small if the body is thin, and so the work of the present paper is relevant to the correct description of the ultimate steady motion. The conclusions of the paper for all except small m do not absolutely contradict any of the features of the Sears & Resler theory but the way in which disturbances propagate to infinity in finite form make it unlikely to be realistic. The solution when m is small, however, contradicts the conclusions of their theory in so many ways, of which the most important are: (i) the conditions at infinity are disturbed, (ii) the fluid velocity is either reduced to rest relative to the body or is undisturbed, (iii) the magnetic field consists of a harmonic element and a piecewise constant element, (iv) the body experiences a drag, that when $m \ll 1$ their theory must be rejected for a steady motion which is set up by the unsteady process stated above. Further, in view of the doubts expressed above its relevance must be queried when m is not small and in particular when $m < 1$.

A very recent paper by Greenspan & Carrier (1959) provides some confirmation of this view. They consider the hydromagnetic boundary layer on a fixed semi-infinite flat plate defined by $x \geqslant 0$, $y = 0$; the fluid is highly conducting and almost inviscid and the conditions at infinity are exactly the same as in the present paper. Their main conclusion is that the boundary layer exists in its conventional form only if $m > 1$. If $m < 1$ it leads to a contradiction suggesting that the problem is incorrectly posed, which is in accord with our view that the flow at infinity upstream is disturbed by the presence of the plate.

We emphasize that these doubts refer to the description of the ultimate state of the motion of the fluid if it is either started from rest or, more generally, if the velocity of the fluid and the magnetic field are each almost uniform at some stage before the steady state is reached. For a steady state achieved without satisfying these conditions the objections do not apply.

REFERENCES

GREENSPAN, H. P. & CARRIER, G. F. 1959 *J. Fluid Mech.* **6**, 77.
LUDFORD, G. S. S. 1959 *Archive Rational Mech. Anal.* **3**, 14.
SEARS, W. R. & RESLER, E. L. 1959 *J. Fluid Mech.* **5**, 257.
STEWARTSON, K. 1956 *Proc. Camb. Phil. Soc.* **52**, 301.
STEWARTSON, K. 1957 *Proc. Camb. Phil. Soc.* **53**, 774.